Equivalence of Boolean Algebras and Pre $A^*$-Algebras

K. Suguna Rao, P. Koteswara Rao

1 Dept. Of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522510, A.P.
2 Dept. Of Commerce, Acharya Nagarjuna University, Nagarjuna Nagar-522510, A.P.

ABSTRACT: In this paper we present definition of Boolean algebra and Boolean sub algebra, examples, theorems on Boolean algebras. Definition of pre $A^*$-algebras, examples and we show that Boolean algebras generates Pre-$A^*$-algebras, correspondence between Boolean algebras and Pre-$A^*$-algebras.

KEY WORDS: Boolean algebra, Pre-$A^*$-algebra.

I. INTRODUCTION:
E. G. MANES introduced an Ada based on C-Algebras introduced by Fernando Guzman and Craig C. Squir. P. Koteswara Rao introduced the concept of $A^*$-Algebras analogous to the E. G. Manes ,Adas. J. Venkateswara Rao introduced the concept of Pre $A^*$-Algebra analogous to the C-Algebra, as a reduct of $A^*$-Algebra. He studied Pre $A^*$-Algebras and their sub directly irreducible representations. It was established that $2 = \{0; 1\}$ and $3 = \{0; 1; 2\}$ are the only sub directly irreducible Pre $A^*$-Algebras and that every Pre $A^*$-Algebra can be embedded in $3^x$, for some set $x$. Also proved that a Pre $A^*$-Algebra can be made into an $A^*$-Algebra by imposing one binary operation and one unary operation and he obtained a sufficient condition for a Pre $A^*$-Algebra to become an $A^*$-Algebra. In this paper we studied Boolean Algebras generates Pre $A^*$-Algebras and One-one correspondence between Boolean algebras and Pre $A^*$-Algebras.

II. BOOLEAN ALGEBRA:
2.1. Definition: An algebra $(B, \wedge, \vee, (\sim), 0, 1)$ is called a Boolean algebra if it satisfies: for every $a, b, c \in B$

i) $a \wedge a = a, a \vee a = a$

ii) $a \wedge b = b \wedge a, a \vee b = b \vee a$

iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

iv) $(a \wedge b) \vee a = a$

v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

vi) $a \wedge 0 = a, a \wedge 1 = a$

vii) $a \wedge a^d = 0, a \wedge a^d = 1$

2.1. Example: $2 = \{0, 1\}$ with $\wedge, \vee, (\sim)$ defined by

<table>
<thead>
<tr>
<th>$\wedge$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\vee$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X$</th>
<th>$X^d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

2.2. Example: Suppose $X$ is a set, $P(X)$ is a Boolean Algebra under the operation of set intersection $\cap$, union $\cup$ and complementation as $(\sim)$, $\emptyset$, $X$ as $\wedge$, $\vee$, $(\sim)$, 0 and 1.

2.3. Example: The class of all logical statements form Boolean algebra under the operations and $(\wedge)$, or $(\vee)$, not $(\sim)$, fallacy $f$ and tautology $t$ as $\wedge$, $\vee$, $(\sim)$, 0 and 1.

2.4. Theorem: Suppose $(B, \wedge, \vee, (\sim), 0, 1)$ is a Boolean Algebra, then
i) \( a \lor b = 1, \ a \land b = 0 \Rightarrow b = a^1 \)

ii) \( a^1 = a \)

iii) \( (a \lor b)^1 = a^1 \land b^1, (a \land b)^1 = a^1 \lor b^1 \)

iv) \( a \land b^1 = 0 \Leftrightarrow a \land b = a \)

v) \( 0^1 = 1, \ 1^1 = 0 \)

vi) \( a \land (a^1 \lor b) = a \land b \)

**Proof:**

i) \[ b = b \lor 0 = b \lor (a \land a^1) = (b \lor a) \land (b \lor a^1) = 1 \land (b \lor a^1) = (b \lor a^1) \]

\[ a^1 = a^1 \lor 0 = a^1 \lor (a \land b) = (a^1 \lor a) \land (a^1 \lor b) = 1 \land (a^1 \lor b) = (a^1 \lor b) = b \land a^1 = b \]

\[ \therefore a = a^1 \]

ii) \[ a^1 \land a^1 = 0, \ a^1 \lor a^1 = 0 \Rightarrow a, a^1 \text{ are inverses of } a^1 \]

\[ \therefore a = a^1 \]

iii) \[ (a \lor b) \land (a^1 \land b^1) = 0 \]

\[ (a \lor b) \lor (a^1 \land b^1) = 1 \]

\[ \therefore (a^1 \lor b^1) = a \land b^1 \]

\[ \therefore (a \land b)^1 = a^1 \lor b^1 \]

iv) \[ \text{Suppose } a \land b^1 = 0 \]

\[ a = a \land 1 = a \land (b \lor b^1) = (a \land b) \lor (a \land b^1) = (a \land b) \lor 0 = (a \land b) \]

\[ \therefore a \land b \]

\[ \text{Suppose } a \land b = a \]

\[ a \land b^1 = 0 \lor (a \land b^1) = (a \land a^1) \lor (a \land b^1) = a \land (a^1 \lor b^1) = a \land (a \land b)^1 = a \land a^1 = 0 \]

\[ \therefore a \land b^1 = 0 \]

\[ \lor 0 \lor 1 = 1, \ 0 \land 1 = 0 \]

\[ \Rightarrow 1^1 = 0 \text{ and } 0^1 = 1 \]

v) \[ a \land (a^1 \lor b) = (a \land a^1) \lor (a \land b) = 0 \lor (a \land b) = a \land b \]

\[ \therefore a \land (a^1 \lor b) = a \land b \]

Similarly \( a \lor (a^1 \land b) = a \lor b \)
III. Pre $A^*$- Algebras:

3.1 Definition: An algebra $(A, \land, \lor, (-)^\sim)$ satisfying

(a) $x\sim x = x$, for all $x \in A$,
(b) $x \land x = x$, for all $x \in A$,
(c) $x \land y = y \land x$, for all $x, y \in A$,
(d) $(x \land y)\sim = x\lor y\sim$, for all $x, y \in A$,
(e) $x \land (y \lor z) = (x \land y) \lor z$, for all $x, y, z \in A$,
(f) $x \land (y \lor z) = (x \lor y) \lor (x \lor z)$, for all $x, y, z \in A$,
(g) $x \land y = x \land (x\lor y)$, for all $x, y, z \in A$,

is called a Pre $A^*$-algebra.

3.2 Example: $\mathbb{3} = \{0, 1, 2\}$ with operations $\land, \lor, (-)^\sim$ defined below is a Pre $A^*$-algebra

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

3.3 Note: $(2, \land, \lor, (-)^\sim)$ is a Boolean algebra. So every Boolean algebra is a Pre $A^*$-Algebra

3.4 Lemma: Let $A$ be a pre $A^*$-Algebra and $a \in A$ be an identity for $\land$, then $a^\sim$ is an identity for $\lor$, a unique if it exists, and is denoted by 1 and $a^\sim$ by 0, i.e.

(i) $a \land x = x$ for all $x \in A$ (ii) $a^\sim \lor x = x$ for all $x \in A$

3.5 Lemma: Let $A$ be a pre $A^*$-Algebra with 1 and 0 and let $x, y \in A$

(i) If $x \lor y = 0$, implies $x = 0$. (ii) If $x \lor y = 1$, implies $x \lor x^\sim = 1$.

3.6. Theorem: Let $(B, \land, \lor, (-)^\sim, 0, 1)$ be a Boolean algebra, then $A(B) = \{(a, b) / a, b \in B, \land = 0\}$ becomes a Pre-$A^*$-algebra, where $\land, \lor, (-)^\sim$ are defined as follows.

For $a = (a_1, a_2), b = (b_1, b_2) \in A(B)$

i) $a \land b = (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$

ii) $a \lor b = (a_1b_1 + a_1b_2 + a_2b_1, a_2b_2)$

iii) $a^\sim = (a_2, a_1)$

iv) $1 = (1,0), 0 = (0,1), 2 = (0,0)$

Proof: Clearly $(a\land b)^\sim = a\lor b$

(i) $(a^\sim)^\sim = (a_2, a_1)^\sim$

= $(a_1, a_2)$

= $(a^\sim)^\sim = a$

(ii) $a \land a = a$

Now $a \land a = (a_1, a_2) \land (a_1, a_2)$

= $(a_1a_1, a_1a_2 + a_1a_2 + a_2a_2)$

= $(a_1, 2a_1a_2 + a_2)$

= $(a_1, a_2)$

= $(a, a) = a$

= $(a^\sim)^\sim = a$
Let $a(x) = (a_1, a_2) \Lambda (b_1, b_2)$.

Proof:

(i) is clear.

To prove (ii), define $f: A \rightarrow A(B)$ by $f(a) = (a_1, a_2, a_1^\#)$. Let $a = b \Leftrightarrow a_1 = b_1, a_2 = b_2 \Leftrightarrow (a_1, a_2, a_1^\#) = (b_1, b_2, b_1^\#) \Leftrightarrow f(a) = f(b)$.

Let $\alpha \in A(B)$.

$\Rightarrow \alpha = (a_1, a_1^\#), where a_1, a_1^\# \in B(A)$

Let $a(x) = a_1 \ast a_1^\#$

Let $f(a(x)) = f((a_1, a_1^\#))$.

$\Rightarrow f(a) = (a_1, a_2, a_1^\#) \Leftrightarrow f(a) = \alpha$

$\Rightarrow f$ is well defined.

Let $\alpha \in A(B)$.

$\Rightarrow \alpha = (a_1, a_1^\#), where a_1, a_1^\# \in B(A)$

Let $a(x) = a_1 \ast a_1^\#$

Let $f(a(x)) = f((a_1, a_1^\#))$.

$\Rightarrow f(a) = (a_1, a_2, a_1^\#) \Leftrightarrow f(a) = \alpha$

$\Rightarrow f$ is onto.

$\therefore f(a \land b) = f(a \land b_1)$

$\therefore f(a \land b) = f(a) \lor f(b)$
Let $A_1, A_2$ be Pre-$A^*$ algebra, $B_1, B_2$ be Boolean algebra.

Then

(i) $A_1 \cong A_2$ iff $B(A_1) \cong B(A_2)$

(ii) $B_1 \cong B_2$ iff $A(B_1) \cong A(B_2)$

**Proof:**

First we prove the following.

(a) $A_1 \cong A_2 \Rightarrow B(A_1) \cong B(A_2)$

(b) $B_1 \cong B_2 \Rightarrow A(B_1) \cong A(B_2)$

(a) $A_1 \cong A_2$

Let: $f: A_1 \rightarrow A_2$ be a Pre-$A^*$ algebra isomorphism

Let $a \in B(A_1) \Rightarrow \exists x \in A_1 \exists y = x_\pi$

$f(a) = f(x_\pi) \in B(A_2)$

Let $b \in B(A_2) \Rightarrow \exists y \in A_2 \exists b = y_\pi$

$
\exists f: A_1 \rightarrow A_2$ is isomorphism and $y \in A_2$

$
\Rightarrow \exists x \in A_1 \exists f(x) = y$

$f(x_\pi) = (f(x))_\pi = y_\pi = b$

$
\exists a \in B(A_1) \iff f(a) \in B(A_2)$

$\exists: f: B(A_1) \rightarrow B(A_2)$ is a Boolean isomorphism

$\exists B(B_1) \cong B(B_2)$

(b) Suppose $B_1 \cong B_2$

Let: $f: B_1 \rightarrow B_2$ be a Boolean isomorphism

Let $a, b \in B_1, a \land b = 0 \Rightarrow f(a) \land f(b) = 0$

Define $g: A(B_1) \rightarrow A(B_2)$ as follows

Let $(a, b) \in A(B_1) \Rightarrow (a, b) \in B_1, a \land b = 0$

$g(a, b) = f(a), f(b) \in A(B_2)$

$\vdash g$ is well defined and $g$ is bisection

$g[(a, b)] \land (x, y) = g(ax, ay + bx + by) = [f(ax), f(ay + bx + by)]$

$= [f(a)f(x), f(a)f(y) + f(b)f(x) + f(b)f(y)]$

$= (f(a), f(b)) \land (f(x), f(y))$

$g[(a, b)] = g(a, b) \land g(x, y)$

$g[(a, b)'] = g(b, a) = (f(b), f(a))$

$= (f(a), f(b))$

$= (g(a, b))$

$\vdash g[(a, b')] = (g(a, b))$

$g[(a, b)] \lor (x, y) = g(ax + ay + bx, by) = [f(ax + ay + bx), f(by)]$

$= [f(a)f(x) + f(a)f(y) + f(b)f(x), f(b)f(y)]$

$= (f(a), f(b)) \lor (f(x), f(y))$

$= g(a, b) \lor g(x, y)$

$\vdash g[(a, b)] \lor (x, y) = g(a, b) \lor g(x, y)$
Equivalence Of Boolean Algebras...

\[ \therefore A \left( B_1 \right) \cong A \left( B_2 \right) \]

(i) From (a) \( A_1 \cong A_2 \Rightarrow B \left( A_1 \right) \cong B \left( A_2 \right) \)

Suppose \( B \left( A_1 \right) \cong B \left( A_2 \right) \)

\[ \Rightarrow A \left( B(A_1) \right) \cong A \left( B(A_2) \right) \] by (b)

But \( A \left( B(A_1) \right) \cong A_1 \) by 2.2 by (ii)

\[ A \left( B(A_2) \right) \cong A_2 \]

\[ \therefore A_1 \cong A_2 \]

(ii) From (b) \( B_1 \cong B_2 \Rightarrow A \left( B_1 \right) \cong A \left( B_2 \right) \)

Suppose \( A \left( B_1 \right) \cong A \left( B_2 \right) \)

\[ \Rightarrow B \left( A \left( B_1 \right) \right) \cong B \left( A \left( B_2 \right) \right) \]

But \( B \left( A \left( B_1 \right) \right) \cong B_1 \)

\[ B \left( A \left( B_2 \right) \right) \cong B_2 \]

\[ \therefore B_1 \cong B_2 \]

REFERENCES:


